

## COMMON FIXED POINT THEOREM IN S FUZZY METRIC SPACES

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### ABSTRACT

Our aim of this paper is to obtain a common fixed point theorem for four self mappings of generalized metric space for integral type.

**KEYWORDS:** Self Mapping, Complete Metric Spaces, t-norm, Common Fixed Point Theorem.

**SUBJECT CLASSIFICATION:** 47H10, 54H25.

### 1. INTRODUCTION

For proposing non-additive models of uncertainty, the most radical attempt was initiated by Zadeh[9] in 1965 with the publication of his paper “Fuzzy Sets”. The notion of fuzzy set is a turning point in the development of mathematics. Consequently the last three decades were very productive of mathematics. Fuzzy fixed point theory has become an area of interest for specialists in fixed point theory.

In this paper we establish a general common fixed point theorems, which generalize the result of Singh and Chouhan[3,4], Singh, B. and Sharma[5,6,7], Som, T and Mukherjee Ray[8].

In 2002, an analogue of a Banach contraction principle for integral type inequality Branciari [1] obtained a fixed point theorem for a single mapping. After the paper of Branciari, a lot of research works have been carried out on generalizing contractive conditions of integral type for different contractive mappings satisfying various known properties.

### 2. PRELIMINARIES

**Theorem 2.1:** (Branciari) [1] Let  $(X, d)$  be a complete metric space,  $c \in (0,1)$  and let  $f: X \rightarrow X$  be a mapping such that for each  $x, y \in X$

$$\int_0^{d(fx, fy)} \varphi(t) dt \leq c \int_0^{d(x, y)} \varphi(t) dt$$

Where  $\varphi: [0, +\infty) \rightarrow [0, +\infty)$  is a Lebesgue – integrable mapping which is summable on each compact subset of  $[0, +\infty)$ , nonnegative and such that for each  $\varepsilon > 0$ ,  $\int_0^\varepsilon \varphi(t) > 0$ , then  $f$  has a unique fixed point  $a \in X$  such that for each  $x \in X$ ,  $\lim_{n \rightarrow \infty} f^n x = a$ .

**Theorem 2.2:** [2] Let  $(X, d)$  be a complete metric space and  $f: X \rightarrow X$  such that

$$\int_0^{d(fx, fy)} u(t) dt \leq \alpha \int_0^{d(x, fx) + d(y, fy)} u(t) dt + \beta \int_0^{d(x, y)} u(t) dt + \gamma \int_0^{\max\{d(x, fy), d(y, fy)\}} u(t) dt,$$

For each  $x, y \in X$  with non-negative real's  $\alpha, \beta, \gamma$  such that  $2\alpha + \beta + 2\gamma < 1$ , where  $u: [0, +\infty) \rightarrow [0, +\infty)$  is a

Lesbesgue – integrable mapping which is summable, non-negative and such that for each  $\epsilon > 0$ ,  $\int_0^\epsilon u(t)dt > 0$ . then f has a unique fixed point in X.

There is a gap in the proof of Theorem 2.2 In fact, the authors [2]

Used the inequality  $\int_0^{a+b} u(t) \leq \int_0^a u(t)dt + \int_0^b u(t)dt$  for  $0 \leq a < b$ , which is not true in general. The aim of the paper is to present in the presence of this inequality an extension of Theorem 2.2 using altering distance functions.

On taking the concept of Branciari we establish common fixed point theorem in S fuzzy space for four integral type mapping.

**Definition 2.3:** The 3-tuple  $(X, S, *)$  is said to be a S-Fuzzy Metric Space if X is an arbitrary Set \* is a continuous t-norm and S is a Fuzzy set on  $X^3 \times (0, \infty)$  Satisfying the following conditions.

- (i)  $S(x, y, z, t) > 0$
- (ii)  $S(x, y, z, t) = 1$  if and only if  $x = y = z$
- (iii)  $S(x, y, z, t) = S(y, z, x, t) = S(z, y, x, t)$  (Symmetry)
- (iv)  $S(x, y, z, r + s + t) \geq S(x, y, w, r) * S(x, w, z, s) * S(w, y, z, t)$  (Tetrahedral inequality)
- (v)  $S(x, y, z, .): (0, \infty) \rightarrow [0, 1]$  is continuous for all  $x, y, z, w \in X$  and  $r, s, t > 0$

Geometrically  $S(x, y, z, t)$  represents the Fuzzy Perimeter of the triangle whose vertices are the points x, y and z with respect tot  $t > 0$

### 3. MAIN RESULTS

**Theorem 3.1:** Let A, B, E and F be self mappings of a complete fuzzy metric space  $(X, S, *)$  with continuous t-norm \* defined by  $a * b = \min\{(a, b) : a, b \in [0, 1]\}$  satisfying the following conditions

$$A(x) \subset F(x), B(x) \subset E(x) \text{ For all } x, y \in X, t > 0 \tag{3.1.1}$$

$$\text{Mappings E and F are continuous} \tag{3.1.2}$$

$$\{A, E\} \text{ and } \{B, F\} \text{ are weakly mapping} \tag{3.1.3}$$

$$\text{For all } x, y \in X, t > 0, k \in [0, 1]$$

$$\int_0^{S(Ax, By, kt)} \varphi(t)dt \geq \int_0^J \varphi(t)dt \tag{3.1.4}$$

$$\text{Where } J = \min \{S(Ax, Ex, t), S(By, Fy, t), S(Ex, Fy, t),$$

$$\frac{S(Ax, Ex, 2t)S(By, Fy, 2t)}{S(Ex, Fy, t)}, \frac{S(Bx, Ex, 2t)}{S(Ex, Fy, t)},$$

$$\frac{S(Bx, Fy, t) + S(By, Ex, t)}{2}\}$$

$$\text{For all } x, y \in X, \lim S(x, y, t) \rightarrow 1 \text{ as } t \rightarrow \infty \tag{3.1.5}$$

Then A, B, E and F have a unique common fixed point in X.

**Proof:** Let  $x_0 \in X$  be arbitrary. Since  $A(x) \subseteq F(x)$ . We can find a point  $x_1$  in  $X$  such that  $Ax_0 = Fx_1$ . Also since  $B(x) \subseteq E(x)$  We can choose a point  $x_2$  with  $Bx_1 = Ex_2$ .

Using this argument repeatedly we can construct sequence  $\{y_n\}$  in  $X$  such that  $y_{2n-1} = Fx_{2n-1} = Bx_{2n-2}$  and  $y_{2n} = Ex_{2n} = Bx_{2n-1}$ ,  $n = 1, 2, 3, \dots$

From (3.1.4) we have

$$\int_0^{S(y_{2n+1}, y_{2n+2}, kt)} \varphi(t) dt = \int_0^{S(Ax_{2n}, Bx_{2n+1}, kt)} \varphi(t) dt \geq \int_0^{J_1} \varphi(t) dt$$

Where

$$J_1 = \min \{S(Ax_{2n}, Ex_{2n}, t), S(Bx_{2n+1}, Fx_{2n+1}, t), S(Ex_{2n}, Fx_{2n+1}, t), \frac{S(Ax_{2n}, Ex_{2n}, 2t)S(Bx_{2n+1}, Fx_{2n+1}, 2t)}{S(Ex_{2n}, Fx_{2n+1}, t)}, \frac{S(Ax_{2n}, Ex_{2n}, 2t)}{S(Ex_{2n}, Fx_{2n+1}, t)}, \frac{S(Ax_{2n}, Fx_{2n+1}, t) + S(Bx_{2n+1}, Ex_{2n}, t)}{2}\}$$

$$\geq \int_0^{J_2} \varphi(t) dt$$

Where

$$J_2 = \min \{S(y_{2n}, y_{2n+1}, t), S(y_{2n+2}, y_{2n+1}, t), S(y_{2n}, y_{2n+1}, t), \frac{S(y_{2n+1}, y_{2n+2}, 2t)S(y_{2n+2}, y_{2n+1}, 2t)}{S(y_{2n}, y_{2n+1}, t)}, \frac{S(y_{2n+1}, y_{2n+2}, 2t)}{S(y_{2n}, y_{2n+1}, t)}, \frac{S(y_{2n+1}, y_{2n+1}, t) + S(y_{2n+2}, y_{2n}, t)}{2}\}$$

$$\int_0^{S(y_{2n+1}, y_{2n+2}, kt)} \varphi(t) dt \geq \int_0^{S(y_{2n}, y_{2n+1}, t)} \varphi(t) dt$$

Which implies in general

$$\int_0^{S(y_n, y_{n+1}, kt)} \varphi(t) dt \geq \int_0^{S(y_{n-1}, y_n, t)} \varphi(t) dt \tag{3.1.6}$$

To prove that  $\{y_n\}$  is a Cauchy Sequence we shall prove

$$S(y_n, y_{n+m}, t) \geq 1 - \lambda \tag{3.1.7}$$

is true for all  $n \geq n_0$  and every  $m \in N$

Here we use induction method from (3.1.6) we have

$$\int_0^{S(y_n, y_{n+1}, t)} \varphi(t) dt \geq \int_0^{S(y_{n-1}, y_n, \frac{t}{k})} \varphi(t) dt \geq \int_0^{S(y_{n-2}, y_{n-1}, \frac{t}{k^2})} \dots \int_0^{S(y_0, y_1, \frac{t}{k^n})} \varphi(t) dt \rightarrow 1 \text{ as } n \rightarrow \infty$$

I.e. for  $t > 0, \lambda \in (0, 1)$  we can choose  $n_0 \in N$  such that  $S(y_n, y_{n+1}, t) \geq 1 - \lambda$

Thus (3.1.7) is true for  $m=1$  supposes (3.1.7) is true for  $m$  then we shall prove that it is also true for  $m+1$ .

Using the definition of fuzzy metric space by (3.1.6) and (3.1.7) we have

$$\int_0^{S(y_n, y_{n+m+1}, t)} \varphi(t) dt \geq \int_0^{\min\{S(y_n, y_{n+m}, t/2), S(y_{n+m}, y_{n+m+1}, t/2)\}} \varphi(t) dt$$

$$\geq 1 - \lambda$$

Hence (3.1.7) is true for  $m+1$ . Thus  $\{y_n\}$  is a Cauchy Sequence. By completeness for  $(X, S, *)$ ,  $\{y_n\}$  converges to some point  $z$  in  $X$ . Thus  $\{Ax_{2n}\}, \{Ex_{2n}\}, \{Bx_{2n-1}\}$  and  $\{Fx_{2n-1}\}$  also converges to  $z$ . Now  $Ax_{2n} \rightarrow z$  and  $E$  is continuous.

Hence  $EAx_{2n} \rightarrow Ez$ .

Thus for  $t > 0, \lambda \in (0, 1)$  there exist an  $n_0 \in N$  such that

$$S(EAx_{2n}, Ez, t/2) \geq 1 - \lambda \text{ for all } n > n_0$$

Using (3.1.3) we have

$$\int_0^{S(AEx_{2n}, EAx_{2n}, t/2)} \varphi(t) dt \rightarrow 1$$

$$\int_0^{S(AEx_{2n}, Ez, t/2)} \varphi(t) dt \geq \int_0^{\min\{S(AEx_{2n}, EAx_{2n}, t/2), S(EAx_{2n}, Ez, t/2)\}} \varphi(t) dt$$

$$\geq 1 - \lambda$$

Hence  $AEx_{2n} \rightarrow Ez$

(3.1.8)

Similarly  $BFx_{2n-1} \rightarrow Fz$

(3.1.9)

Using (3.1.4) we have

$$\int_0^{S(AEx_{2n}, BFx_{2n-1}, kt)} \varphi(t) dt \geq \int_0^{J_3} \varphi(t) dt$$

Where

$$\min\{S(AEx_{2n}, E^2x_{2n}, t), S(BFx_{2n-1}, F^2x_{2n-1}, t),$$

$$S(E^2x_{2n}, F^2x_{2n-1}, t),$$

$$J_3 = \frac{S(AEx_{2n}, E^2x_{2n}, 2t)S(BFx_{2n-1}, F^2x_{2n-1}, 2t)}{S(E^2x_{2n}, F^2x_{2n-1}, t)}, \frac{S(AEx_{2n}, E^2x_{2n}, 2t)}{S(E^2x_{2n}, F^2x_{2n-1}, t)},$$

$$\frac{S(AEx_{2n}, F^2x_{2n-1}, t) + S(BFx_{2n-1}, E^2x_{2n}, t)}{2}\}$$

Taking limit as  $n \rightarrow \infty$  and using (3.1.8) and (3.1.9) we get

$$\int_0^{S(Ez, Fz, kt)} \varphi(t) dt \geq \int_0^{S(Ez, Fz, t)} \varphi(t) dt$$

Which implies that

$$Ez = Fz$$

(3.1.10)

Now

$$\int_0^{S(Az, BFx_{2n-1}, kt)} \varphi(t) dt \geq \int_0^{J_4} \varphi(t) dt$$

Where

$$\min \{S(Az, Ez, t), S(BFx_{2n-1}, F^2x_{2n-1}, t), S(Ez, F^2x_{2n-1}, t), J_4 = \frac{S(Az, Ez, 2t)S(BFx_{2n-1}, F^2x_{2n-1}, 2t)}{S(Ez, F^2x_{2n-1}, t)}, \frac{S(AEz, Ez, 2t)}{S(Ez, F^2x_{2n-1}, t)}, \frac{S(AEz, F^2x_{2n-1}, t) + S(BFx_{2n-1}, Ez, t)}{2}\}$$

Taking the limit as  $n \rightarrow \infty$  and using (3.1.9) and (3.1.10) we get

$$\int_0^{S(Az, Fz, kt)} \varphi(t) dt \geq \int_0^{S(Az, Fz, t)} \varphi(t) dt$$

Which implies that

$$Az = Fz \tag{3.1.11}$$

$$\int_0^{S(Az, Bz, kt)} \varphi(t) dt \geq \int_0^{J_5} \varphi(t) dt$$

Where

$$\min \{S(Az, Ez, t), S(Bz, Fz, t), S(Ez, Fz, t), J_5 = \frac{S(Az, Ez, 2t)S(Bz, Fz, 2t)}{S(Ez, Fz, t)}, \frac{S(AEz, Ez, 2t)}{S(Ez, Fz, t)}, \frac{S(AEz, Fz, t) + S(Bz, Ez, t)}{2}\}$$

$$\int_0^{S(Az, Bz, kt)} \varphi(t) dt \geq \int_0^{J_6} \varphi(t) dt$$

Where

$$\min \{S(Fz, Ez, t), S(Bz, Az, t), S(Ez, Fz, t), J_6 = \frac{S(Az, Ez, 2t)S(Bz, Fz, 2t)}{S(Ez, Fz, t)}, \frac{S(AEz, Ez, 2t)}{S(Ez, Fz, t)}, \frac{S(AEz, Fz, t) + S(Bz, Ez, t)}{2}\}$$

Which implies that

$$Az = Bz \tag{3.1.12}$$

Using (3.1.10) and (3.1.12) we get

$$Az = Bz = Ez = Fz \tag{3.1.13}$$

$$\text{Now } \int_0^{S(Ax_{2n}, Bz, kt)} \varphi(t) dt \geq \int_0^{J_7} \varphi(t) dt$$

Where

$$\min \{S(Ax_{2n}, Ex_{2n}, t), S(Bz, Fz, t), S(Ex_{2n}, Fz, t), J_7 = \frac{S(Ax_{2n}, Ex_{2n}, 2t)S(Bz, Fz, 2t)}{S(Ex_{2n}, Fz, t)}, \frac{S(Ax_{2n}, Ex_{2n}, 2t)}{S(Ex_{2n}, Fz, t)}, \frac{S(Ax_{2n}, Fz, t) + S(Bz, Ex_{2n}, t)}{2}\}$$

$$\geq \int_0^{J_8} \varphi(t) dt$$

Where

$$J_8 = \min \left\{ S(\text{Ex}_{2n}, \text{Fz}, t), S(\text{Ax}_{2n}, \text{Ex}_{2n}, t), S(\text{Bz}, \text{Fz}, t), \right. \\ \left. \frac{S(\text{Ax}_{2n}, \text{Ex}_{2n}, 2t)S(\text{Bz}, \text{Fz}, 2t)}{S(\text{Ex}_{2n}, \text{Fz}, t)}, \right. \\ \left. \frac{S(\text{Ax}_{2n}, \text{Ex}_{2n}, 2t)}{S(\text{Ex}_{2n}, \text{Fz}, t)}, \frac{S(\text{Ax}_{2n}, \text{Fz}, t) + S(\text{Bz}, \text{Ex}_{2n}, t)}{2} \right\}$$

Taking limit as  $n \rightarrow \infty$  and using (3.1.13) we get

$$\int_0^{S(z, \text{Bz}, kt)} \varphi(t) dt \geq \int_0^{S(z, \text{Bz}, t)} \varphi(t) dt$$

Which implies that

$$Z = \text{Bz} \tag{3.1.14}$$

Thus  $z$  is a common fixed point of  $A, B, E$  and  $F$ . For the uniqueness let  $w$  be another common fixed point of said mappings. Then from (3.1.4)

$$\int_0^{S(\text{Az}, \text{Bw}, kt)} \varphi(t) dt \geq \int_0^{J_9} \varphi(t) dt$$

Where  $J_9 = \min \{ S(\text{Az}, \text{Ez}, t), S(\text{Bw}, \text{Fw}, t), S(\text{Ez}, \text{Fw}, t),$

$$\left. \frac{S(\text{Az}, \text{Ez}, 2t)S(\text{Bw}, \text{Fw}, 2t)}{S(\text{Ez}, \text{Fw}, t)}, \frac{S(\text{Az}, \text{Ez}, 2t)}{S(\text{Ez}, \text{Fw}, t)}, \frac{S(\text{Az}, \text{Fw}, t) + S(\text{Bw}, \text{Ez}, t)}{2} \right\}$$

$$\text{i. e. } \int_0^{S(z, \text{w}, kt)} \varphi(t) dt \geq \int_0^{S(z, \text{w}, t)} \varphi(t) dt$$

Hence

$$z = w$$

This completes the proof

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